Midsemestral examination 2009
M.Math. IInd year
Algebraic Number Theory : B.Sury

Instructions :
(0) *This is an open-book test with ONE book (other than Problem books) chosen by each of you before entering the exam hall. Each of you may choose a different book if you want to !*
(i) *Answer any 3 questions from the global part G1 to G5 and any 3 from the local part L1 to L5.*
(ii) *The notation ζ_n is used for any primitive n-th root of unity.*
(iii) *You may use some standard results but quote them precisely. Be Brief !*

G 1.
Let $K = \mathbf{Q}(\theta)$ where $\theta^5 = \theta + 1$. Show that $X^5 - X - 1$ is irreducible over $\mathbf{Q}$ and that $\mathbf{O}_K = \mathbf{Z}[\theta]$.
Hint : Look at the polynomial mod 5; you may also use the fact that the discriminant is $(-1)^{\binom{n}{2}}((n^n b^{n-1} + (-1)^{n+1}(n-1)^{n-1}a^n)$ for an irreducible polynomial of the form $X^n + aX + b$.

G 2.
Determine the factorization of 31 in the ring of integers of $\mathbf{Q}(2^{1/3})$.

G 3.
Prove (briefly) that only finitely many primes ramify in the ring of integers of any given number field.

G 4.
Prove that $\mathbf{Q}(\zeta_{23})$ has class number > 1.
Hint : You may use the fact that $\mathbf{Q}(\sqrt{-23}) \subset \mathbf{Q}(\zeta_{23})$.

OR

Determine the class group of $\mathbf{Q}(\sqrt{-6})$.

G 5.
Find (with proof) the fundamental unit of $\mathbf{Q}(\sqrt{13})$.

L 1.
Prove that the set of equations

$$10a^2 + 34c^2 = 17u^2 + v^2$$

$$a^2 + 5c^2 = uv$$

do not have a common solution in $\mathbf{Q}_{17}$.

L 2.
Use Hensel's lemma to prove that $a \in \mathbf{Z}_3^*$ is a cube if and only if $a \equiv \pm 1 \bmod 9\mathbf{Z}_3$.

L 3.
Show that $\mathbf{Q}_p(\zeta_{p^n})$ has degree $\phi(p^n)$ over $\mathbf{Q}_p$.

OR

Find the radius of convergence of the exponential series $exp(x)$ over $\mathbf{Q}_p$.

L 4.
Prove that a totally ramified extension of a p-adic field k is given by adjoining the root of an Eisenstein polynomial over k.

L 5.
Let K denote an algebraic closure of $\mathbf{Q}_p$ and $\mathcal{O}$ denote the integral closure of $\mathbf{Z}_p$ in K. Prove that $\mathcal{O}$ is not Noetherian.
Hint : Show that $\mathcal{O}$ has only one nonzero prime ideal and it is not finitely generated.

OR

Let K denote an algebraic closure of $\mathbf{Q}_p$. Let $\{b_n\}$ be a sequence of roots of unity of order coprime to p in K satisfying the following properties :
$b_1 = 1, b_n \in \mathbf{Q}_p(b_{n+1}), [\mathbf{Q}_p(b_{n+1}) : \mathbf{Q}_p(b_n)] > n$.
Deduce that K is not complete by showing that the series $\sum_n b_n p^n$ does not converge in K.